SOME DOMINATING SETS OF LEXICOGRAPHIC PRODUCT GRAPHS OF EULER TOTIENT CAYLEY GRAPHS WITH ARITHMETIC V_n GRAPHS

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Abstract: The theory of domination in graphs is an emerging area of research in graph theory today. It has been studied extensively and finds applications to various branches of Science & Technology.Products are often viewed as a convenient language with which one can describe structures, but they are increasingly being applied in more substantial ways. In this paper, we consider lexicographic product graph of Euler totient Cayley graphs with Arithmetic V_n graphs and present some results on domination parameters of these graphs.

Index Terms: Dominating set, total dominating set, Euler totient Cayley graph, Arithmetic V_n graph and lexicographic product graph.

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1. INTRODUCTION

Number Theory is one of the oldest branches of mathematics, which inherited rich contributions from almost all greatest mathematicians, ancient and modern.

Nathanson [1] paved the way for the emergence of a new class of graphs, namely Arithmetic Graphs by introducing the concepts of Number Theory, particularly, the Theory of Congruences in Graph Theory. Cayley graphs are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an Arithmetic graph. Inspired by the interplay between Number Theory and Graph Theory several researchers in recent times are carrying out extensive studies on various Arithmetic graphs in which adjacency between vertices is defined through various arithmetic functions.

Domination in graphs has been an extensively research branch of graph theory. (For more details refer Cockayne [2] and [3]). Dominating sets play an important role in practical applications, such as allocation of re-hydrants or serving sites of other supplies, modeling of relations in human groups or animal biotopes. They have applications in diverse areas such as logistics and networks design, mobile computing, resource allocation and telecommunication. Cayley graphs are excellent models for interconnection networks, investigated in connection with parallel processing and distributed computation.

In this section we present necessary definitions, observations and some useful results that we need for next sections.

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2. DOMINATING SET

A subset D of $V(G)$ is said to be a dominating set of G if every vertex in $V - D$ is adjacent to a vertex in D.

The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$.

3. TOTAL DOMINATING SET

Let G be a graph without isolated vertices. Then a total dominating set T is a subset of V such that every vertex of V is adjacent to some vertex in T.

The minimum cardinality of a total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(G)$.

4. LEXICOGRAPHIC PRODUCT GRAPHS

The lexicographic product was first studied by Felix Hausdorff in the year 1914. Later this product was introduced as the composition of graphs by Harary in the year 1959. There has been a rapid growth of research on the structure of this product and their algebraic settings, after the publication of the paper, on the group of the composition of two graph by Haray, F. [4]. Geller, D and Stahl [5]   determined the chromatic number and other functions of this product in the year 1975.

Feigenbaum and Schaffer [6] carried their research on the problem of recognizing whether a graph is a lexicographic product is equivalent to the graph isomorphism problem in the year 1986. Imrich and Klavzar [7] discussed the automorphisms, factorizations and non-uniqueness of this product.

This product is in general non-commutative. But two graphs G and H commute with respect to the lexicographic product if G and H are complete or if both are totally disconnected graphs (For a detailed   description, refer Imrich and Klavzar [7])

The Lexicographic Product, $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph G such that $V(G) = V(G_1) \times V(G_2)$ and any two distinct vertices (u_1, v_1) and (u_2, v_2) of G are adjacent if

either $u_1 u_2$ is an edge of G_1 or $u_1 = u_2$ and $v_1 v_2$ is an edge of G_2.

Now we consider the lexicographic product graph of Euler totient Cayley graphs with Arithmetic V_n graphs. The properties of these graphs are presented in [8]. We briefly present Euler totient Cayley graph and Arithmetic V_n graph.

5. EULER TOTIENT CAYLEY GRAPH

Madhavi [9] introduced the concept of Euler totient Cayley graphs and studied some of its properties. For any positive integer n, let $Z_n = \{0,1,2,....,n-1\}$. Then $(Z_n, \oplus)$, where $\oplus$ is addition modulo is an abelian group of order n. The number of positive integers less than n and relatively prime to n is denoted by $\varphi(n)$ and is called Euler totient function.
Let S denote the set of all positive integers less than n and relatively prime to n
That is $S = \{r \ / \ 1 \le r < n \text{ and } GCD\ (r,n) = 1\}$
Then $|S| = \varphi(n)$.
Now we define Euler totient Cayley graph as follows.
For each positive integer n , let Z_n be the additive group of integers modulo n and S be the set of all numbers less than n and relatively prime to n. The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is   given by $Z_n = \{0,1,2,.....n-1\}$ and the edge set is given by $E = \{(x,y)/x - y \in S \text{   or   } y - x \in S\}$.

The domination parameters of these graphs are studied by Uma Maheswari [8] and we present some of the results which we need without proofs and can be found in [10].

Theorem 5.1: *If n is a prime, then the domination number of $G(Z_n, \varphi)$ is 1.*

Theorem 5.2: *If n is power of a prime, then the domination number of $G(Z_n, \varphi)$ is 2.*

Theorem 5.3: *The domination number of $G(Z_n, \varphi)$ is 2, if $n = 2p$ where p is an odd prime.*

Theorem 5.4:   *Suppose n is neither a prime nor $2p$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ....p_k^{\alpha_k}$, where p_1, p_2 , ... p_k are primes and $\alpha_1, \alpha_2,.... \alpha_k$ are integers $\ge$ 1. Then the domination number of $G(Z_n, \varphi)$ is given by $\gamma\big(G(Z_n, \varphi)\big) = \lambda + 1$, where λ is the length of the longest stretch of consecutive integers in V, each of which shares a prime factor with n.*

Theorem 5.5: *If n is a prime, then the total domination number of $G(Z_n, \varphi)$ is 2.*

Theorem 5.6: *If n is power of a prime, then the total domination number of $G(Z_n, \varphi)$ is 2.*

Theorem 5.7: *The total domination number of $G(Z_n, \varphi)$ is 4, if $n = 2p$, where p is an odd prime.*

Theorem 5.8:   *Suppose n is neither a prime nor $2p$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ....p_k^{\alpha_k}$ where p_1, p_2 , ... p_k are primes and $\alpha_1, \alpha_2,.... \alpha_k$ are integers $\ge$ 1. Then the total domination number of $G(Z_n, \varphi)$ is given by $\gamma_t\big(G(Z_n, \varphi)\big) = \lambda + 1$, where λ is the length of the longest stretch of consecutive integers in V each of which shares a prime factor with n.*

6. ARITHMETIC V_n GRAPH

Vasumathi [11] introduced the concept of Arithmetic V_n graphs and studied some of its properties.
Let n be   a   positive   integer   such   that $n = p_1^{\alpha_1} p_2^{\alpha_2} ....p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in V_n graph if and only if GCD $(u, v) = p_i$ for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex. Hence we consider Arithmetic graph without vertex 1 as the contribution of this isolated vertex is nothing when the properties of these graphs and enumeration of some domination parameters are studied.

Clearly, V_n graph is a connected graph. Because if n is a prime, then V_n graph consists of a single vertex. Hence it is connected. In other cases, by the definition of adjacency in V_n, there exist edges between prime number vertices and their prime power vertices and also to their prime product vertices. Therefore each vertex of V_n is connected to some vertex in V_n.

The domination parameters of these graphs are studied by S.Uma Maheswari [8] and we present some of the results which we need without proofs and can be found in [12].

Theorem 6.1: *If $n = p_1^{\alpha_1} p_2^{\alpha_2} ....p_k^{\alpha_k}$, where p_1, p_2 , ... p_k are primes and $\alpha_1, \alpha_2,.... \alpha_k$ are integers ≥ 1, then the domination number of $G(V_n)$ is given by*

$$\gamma\big(G(V_n)\big) = \begin{cases} k-1 & \text{if } \alpha_i = 1 \text{ for more than one } i \\ k & \text{Otherwise.} \end{cases}$$

where k is the core of n

Theorem 6.2: *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ....p_k^{\alpha_k}$ where $\alpha_i \ge 1, \forall i$. Then the total domination number of $G(V_n)$ is k, where k is the core of n*

7. DOMINATION IN LEXICOGRAPHIC PRODUCT GRAPH $G_1 \circ G_2$

In this section we discuss the dominating sets of the lexicographic product graph of Euler totient Cayley graph G_1 and Arithmetic V_n graph G_2.

Theorem 7.1: *If n is a prime, then the domination number of $G_1 \circ G_2$ is 1.*

Proof: If n is a prime, then $G_1 \circ G_2$ is a complete graph. Hence each single vertex set in $G_1 \circ G_2$ constitute a minimum dominating set. Therefore domination number of $G_1 \circ G_2$ is 1, if n is a prime. ∎

Theorem 7.2 : *The domination number of $G_1 \circ G_2$ is 2, if $n = 2p$ where p is an odd prime.*

Proof: Let $n = 2p$, p is an odd prime. Consider the graph $G_1 \circ G_2$.

Let $V(G_1) = \{0, 1, 2, \dots \dots, 2p-1\} = V_1$
$V(G_2) = \{2, p, 2p\} = V_2$ and
$V(G_1 \circ G_2) = V_1 \times V_2 = V$

be the sets of vertices of the graphs G_1, G_2 and $G_1 \circ G_2$ respectively. By Theorem 1.3, we know that $\gamma(G_1) = 2$. Without loss of generality we may take a minimum dominating set of G_1 as $D_1 = \{u_{d_1}, u_{d_2}\}$ where $|u_{d_1} - u_{d_2}| = p$.

It is obvious that $\{2p\}$ is a dominating set of G_2.

Consider $D = D_1 \times \{2p\} = \{(u_{d_1}, 2p), (u_{d_2}, 2p)\}$. We claim that D is a dominating set of $G_1 \circ G_2$.

Let $(u, v) \in V - D$.
The following cases arise.

Case 1: Suppose $u = u_{d_1}$, $v = 2$ or p. Then by the definition of lexicographic product, vertices $(u_{d_1}, 2)$ and (u_{d_1}, p) in $V - D$ are adjacent to $(u_{d_1}, 2p)$, because 2 and p are adjacent to $2p$ as GCD $(2, 2p) = 2$ and $(p, 2p) = p$.

Case 2: Suppose $u = u_{d_2}$, $v = 2$ or p. By the similar argument as in case 1, vertices $(u_{d_2}, 2)$ and (u_{d_2}, p) in $V - D$ are adjacent to $(u_{d_2}, 2p)$.

Case 3: Suppose $u \neq u_{d_1}$ and $u \neq u_{d_2}$, $v = 2$ or p or $2p$. Since D_1 is a dominating set of G_1, u is adjacent to either u_{d_1} or u_{d_2}, say u_{d_1}. Then by the definition of lexicographic product, (u, v) is adjacent to $(u_{d_1}, 2)$, (u_{d_1}, p) and $(u_{d_1}, 2p)$. Thus (u, v) is adjacent to $(u_{d_1}, 2p)$.

Thus (u, v) in $V - D$ is dominated by at least one vertex in D. Therefore D becomes a dominating set.

Further deletion of any vertex from D does not make D as a dominating set. Suppose we delete a vertex, say $(u_{d_1}, 2p)$ from D. Then vertices $(u_{d_1}, 2)$ and (u_{d_1}, p) are not dominated by the remaining vertex $(u_{d_2}, 2p)$. This is because $u_{d_1} \neq u_{d_2}$ and u_{d_1} is not adjacent to u_{d_2} as $|u_{d_1} - u_{d_2}| = p$. Similar is the case if we delete vertex $(u_{d_2}, 2p)$ from D. Thus D becomes a minimum dominating set.

Thus $\gamma(G_1 \circ G_2) = |D| = 2$. ∎

Theorem 7.3: *If n is neither a prime nor 2p then the domination number of $G_1 \circ G_2$ is*

$\lambda + 1$, *where λ is the length of the longest stretch of consecutive integers in V_1 of G_1 each of which shares a prime factor with n.*

Proof: Suppose n is neither a prime nor $2p$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots p_k^{\alpha_k}$, where $\alpha_i \geq 1$. Let $V_1 = \{0, 1, 2, \dots, n-1\}$ and $V_2 = \{v_1, v_2, \dots, v_m\}$ be the vertex sets of the graphs G_1 and G_2 respectively.

Recall a dominating set of G_1 with minimum cardinality $\lambda + 1$ which is given in Theorem 1.4 by $D_1 = \{u_{d_1}, u_{d_2}, \dots \dots, u_{d_{\lambda+1}}\}$ where $u_{d_1}, u_{d_2}, \dots \dots, u_{d_{\lambda+1}}$ are consecutive integers.

Let $D = D_1 \times v_x$ where v_x be any vertex in V_2 of G_2. Then $D = \{(u_{d_1}, v_x), (u_{d_2}, v_x), \dots \dots, (u_{d_{\lambda+1}}, v_x)\}$

We now claim that D is a dominating set of $G_1 \circ G_2$. Let $(u, v) \in V - D$.

Case 1: Suppose $u = u_{d_i}$ for $i = 1, 2, 3, \dots, \lambda + 1$. Then $(u, v) = (u_{d_i}, v)$ where $1 \leq i \leq \lambda + 1$ and $v \in V_2$ and $v \neq v_x$. Since $u_{d_1}, u_{d_2}, \dots \dots, u_{d_{\lambda+1}}$ are consecutive integers, each u_{d_i} is adjacent to $u_{d_{i+1}}$ for $i = 1, 2, \dots, \lambda$, because GCD $(u_{d_i} - u_{d_{i+1}}, n) = 1$. Hence by the definition of Lexicographic Product, $(u, v) = (u_{d_i}, v)$ is adjacent to $(u_{d_{i+1}}, v)$ for $i = 1, 2, \dots, \lambda$ in D. In particular for $v = v_x$ in V_2, the vertex $(u, v) = (u_{d_i}, v)$ in $V - D$ is adjacent to (u_{d_i}, v_x) in D.

Case 2: Suppose $u \neq u_{d_i}$ for $i = 1, 2, 3, \dots, \lambda + 1$ and $v \in V_2$. Since D_1 is a dominating set of G_1, the vertex u must be adjacent to at least one of the vertices of D, say u_{d_i}. Since u and u_{d_i} are adjacent, by the definition of lexicographic product the vertex (u, v) is adjacent to the vertex (u_{d_i}, v_q), $\forall v_q \in V_2$. In particular to the vertex (u_{d_i}, v_x) in D. Thus all the vertices in $V - D$ are adjacent to at least one vertex in D and D becomes a dominating set in $G_1 \circ G_2$.

We now show that deletion of any vertex from D does not make D as a dominating set. Suppose we delete a vertex (u_{d_i}, v_x) from D for some i, $1 \leq i \leq \lambda + 1$. Since each vertex in G_1 is of degree $\varphi(n)$, vertex u_{d_i} is adjacent to the vertices, say $u_1, u_2, \dots, u_{\varphi(n)}$ respectively. Then the vertices $(u_1, v_x), (u_2, v_x), \dots, (u_{\varphi(n)}, v_x)$ are all not dominated by other vertices of $D - \{(u_{d_i}, v_x)\}$. If so then $u_1, u_2, \dots, u_{\varphi(n)}$ are also dominated by the other vertices of $D_1 - \{u_{d_i}\}$, which implies that D_1 is not a minimum dominating set of G_1, a contradiction.

Therefore D is a minimum dominating set. Further if we construct a dominating set in any other manner then the order of such a set is bigger than the order of D. This follows from the properties of the prime divisors of a number.

Hence $\gamma(G_1 \circ G_2) = |D| = \lambda + 1$. ∎

8. Total Domination in Lexicographic Product Graph $G_1 o\, G_2$

In this section the results on the total dominating sets of direct product graph are discussed for different values of n.

Theorem 8.1 If n is a prime, then the total domination number of $G_1 o\, G_2$ is 2.

Proof: If n is a prime, then $G_1 o G_2$ is a complete graph. Hence any two vertices in $G_1 o\, G_2$ form a minimum total dominating set of $G_1 o\, G_2$. Thus the total domination number of $G_1 o\, G_2$ is 2. ∎

It is interesting to see that the dominating set given in Theorem 2.3, is also a total dominating set. This is proved in the following.
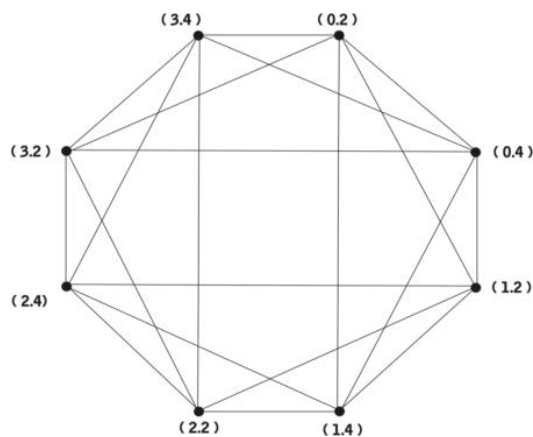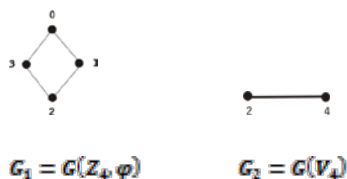
Theorem 8.2: Let $= p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots p_k^{\alpha_k}$, where $\alpha_i \geq 1,$. Then the total domination number of $G_1 o G_2$ is given by $\gamma_t(G_1 o G_2) = \lambda + 1$, where λ is the length of the longest stretch of consecutive integers in V each of which shares a prime factor with n.

Proof: We have proved in Theorem 5.3.3 that the following set $T = \{ (u_{d_1}, v_x), (u_{d_2}, v_x), \ldots\ldots, (u_{d_{\lambda+1}}, v_x) \}$ where v_x is any vertex of G_2 is a dominating set of $G_1 o G_2$. We now show that every vertex of T is adjacent to some other vertex of T, so that T becomes a total dominating set.
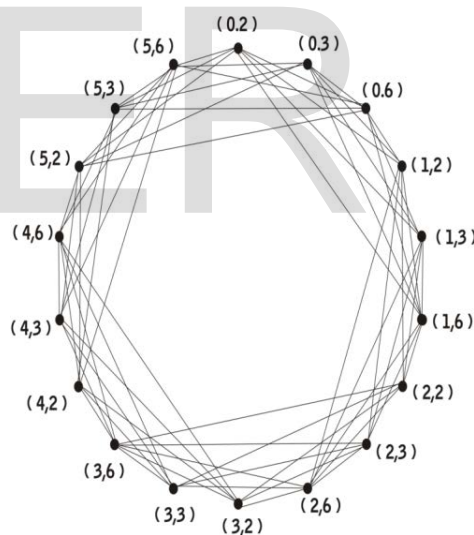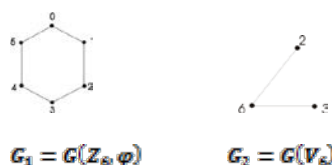
Since $u_{d_1}, u_{d_2}, \ldots\ldots, u_{d_{\lambda+1}}$ are consecutive integers, each u_{d_i} is adjacent to $u_{d_{i+1}}$ for $i = 1, 2, \ldots, \lambda,$ because GCD $(u_{d_i} - u_{d_{i+1}}, n) = 1$. Hence by the definition of Lexicographic Product, each vertex (u_{d_i}, v_x) of T is adjacent to $(u_{d_{i+1}}, v_x)$ of T and vice versa. Thus all the vertices of T are dominated by the vertices of T. Therefore T becomes a total dominating set of $G_1 o G_2$ with minimum cardinality.
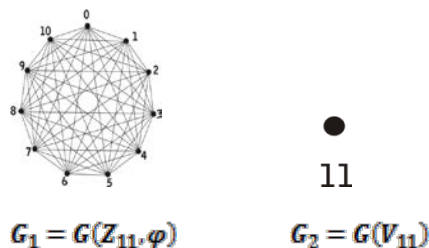
Hence $\gamma_t(G_1 o G_2) = |T| = \lambda + 1$. ∎

9. ILLUSTRATIONS

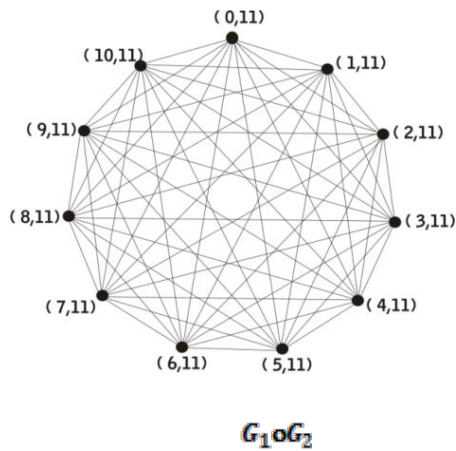$G_1 = G(Z_4, \varphi)$ $G_2 = G(V_4)$

$G_1 o G_2$

$G_1 = G(Z_6, \varphi)$ $G_2 = G(V_6)$

$G_1 o G_2$

$G_1 = G(Z_{11}, \varphi)$ $G_2 = G(V_{11})$

$G_1 o G_2$

Lexicographic product Graph $G_1 o \ G_2$

	Dominating sets	$G_1 = G(Z_n, \varphi)$	$G_2 = G(V_n)$	$G_1 o \ G_2$	Domination Number in $G_1 o \ G_2$
$n = 4$	Minimum Dominating set	$\{0,1\}$	$\{2\}$	$\{(0,2), (1,2)\}$	$\gamma = 2$
	Minimum Total Dominating set	$\{0,1\}$	$\{2,4\}$	$\{(0,2), (1,2)\}$	$\gamma_t = 2$
$n = 6$	Minimum Dominating set	$\{0,3\}$	$\{6\}$	$\{(0,6), (3,6)\}$	$\gamma = 2$
	Minimum Total Dominating set	$\{0,1,2,3\}$	$\{2,6\}$	$\{(0,6), (1,6), (2,6), (3,6)\}$	$\gamma_t = 4$
$n = 11$	Minimum Dominating set	$\{0\}$	$\{11\}$	$\{(0,11)\}$	$\gamma = 1$
	Minimum Total Dominating set	$\{0,1\}$	Does not exist	$\{(0,11), (1,11)\}$	$\gamma_t = 2$

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